B. Sc. Part II (H) 3rd Paper

{Infinite series}

Q. Test the convergence of the series

$$1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots$$

Soln Removal of 1 i.e. first term of the given series will not affect the convergence of the series.

Let the series now be denoted by $\sum U_n$.

$$\therefore \sum U_n = \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots$$

$$\therefore U_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n+1)^2}$$

$$\Rightarrow U_{n+1} = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2 (2n+2)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n+1)^2 (2n+3)^2}$$

$$\therefore \frac{U_n}{U_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \left(\frac{2n+3}{2n+2}\right)^2 = \lim_{n\to\infty} \left[\frac{2+\frac{3}{n}}{2+\frac{2}{n}}\right]^2$$

$$\Rightarrow \lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \left(\frac{2+0}{2+0}\right)^2 = 1$$

So, this test fails.

Now, $$\frac{U_n}{U_{n+1}} - 1 = \frac{(2n+3)^2}{(2n+2)^2} - 1$$

$$= \frac{(2n+3)^2 - (2n+2)^2}{(2n+2)^2}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} - 1 = \frac{(4n+5)}{(2n+2)^2}$$

$$\Rightarrow n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{4n^2+5n}{4n^2+8n+4}$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{4n^2+5n}{4n^2+8n+4} = \lim_{n\to\infty} \frac{4+\frac{5}{n}}{4+\frac{8}{n}+\frac{4}{n^2}}$$

$$= \frac{4+0}{4+0+0} = 1$$

$\Rightarrow$ Raabe's test fails.

Now, $$\left[n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1\right] = \frac{4n^2+5n}{4n^2+8n+4} - 1 = \frac{-3n-4}{(2n+2)^2}$$

$$\Rightarrow \lim_{n\to\infty} \left[n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1\right]\log n = \lim_{n\to\infty} \frac{-(3n+4)\log n}{(2n+2)^2}$$

$$= \lim_{n\to\infty} \frac{\left(3+\frac{4}{n}\right)}{\left(2+\frac{2}{n}\right)^2} \cdot \frac{\log n}{n}$$

$$= \lim_{n\to\infty} \frac{3+0}{(2+0)^2} \cdot \lim_{n\to\infty} \frac{\log n}{n} = \frac{3}{4} \times \lim_{n\to\infty} \left[\frac{d(\log n)}{d(n)}\right]$$

$$= \frac{3}{4} \times \lim_{n\to\infty} \left[\frac{1}{n \times 1}\right] = \frac{3}{4} \times 0 = 0 < 1.$$

Hence, the series is divergent